Common Algebra and Trigonometry Mistakes made in Calculus

The most common comment I get from students struggling in calculus:

"I understand the calculus concepts, but the algebra gets me every time!"

This is a collection of the most common algebra and trigonometry mistakes. The mistakes made are typically easy to correct if the student is willing to go through the explanation and work a few problems, then pledge to never make the mistake again $\ddot\smile$. By far the first three are the most common, but the rest also occur fairly frequently. Students can refer to this handout as needed, and do the practice problems provided to re-solidify their understanding of the concept.

1 The Mad Slasher (TMS)

Some BAD examples of mad slashing are

I. $\dfrac{x+4}{x} = \dfrac{\cancel{x}+4}{\cancel{x}} = 4$

II. $\dfrac{(x-3)x - x^2(x+1)}{x-3} = \dfrac{\cancel{(x-3)}x - x^2(x+1)}{\cancel{x-3}} = x - x^2(x+1)$

III. $\dfrac{\sin^2 x - \cos^2 x}{\sin x} = \dfrac{\sin^{\cancel{2}} x - \cos^2 x}{\cancel{\sin x}} = \sin x - \cos^2 x$

The Mad Slasher is when, in the haste of the moment, start crossing out similar looking expressions on the numerator and denominator. This mistake is understood by realizing that the operation you are doing when cancelling is division; and when you cancel the denominator with a single term you are essentially only dividing that single term by the denominator. The mistake can easily be seen using numbers. For example, it is easy to see that

$$\frac{4+8}{4} = \frac{12}{4} = 3 \quad \text{definitely does not equal} \quad \frac{\cancel{4}+8}{\cancel{4}} = 1+8 = 9.$$

To be able to simplify in this way everything must be factored completely. So for this simple example you would factor first and then simplify

$$\frac{4+8}{4} = \frac{4(1+2)}{4} = \frac{\cancel{4}(1+2)}{\cancel{4}} = 3.$$

which is correct. This works similarly for complicated expressions. Here are some correct examples:

$$\text{I.} \quad \frac{x^2-4x-12}{x-6} = \frac{(x-6)(x+2)}{x-6} = \frac{\cancel{(x-6)}(x+2)}{\cancel{x-6}}.$$

$$\text{II.} \quad \frac{(x+3)(x-1)-x(x-1)}{x-1} = \frac{(x-1)[(x+3)-x]}{x-1} = \frac{\cancel{(x-1)}[(x+3)-x]}{\cancel{x-1}}.$$

$$\text{III.} \quad \frac{\sin^3 x - \sin x \cos^2 x}{\sin x} = \frac{\sin x(\sin^2 x - \cos^2 x)}{\sin x} = \frac{\cancel{\sin x}(\sin^2 x - \cos^2 x)}{\cancel{\sin x}}.$$

Here are some problems for you to try. Simplify the following:

1. $\dfrac{x^2-16}{x^2} \cdot \dfrac{x^2-4x}{x^2-x-12} =$

2. $\dfrac{x^3-27}{x^4-9x^2} \cdot \dfrac{x^5-6x^4+9x^3}{x^2+3x+9} =$

3. $\dfrac{\sin^3 x + 4\sin^2 x - 12\sin x}{\sin^2 x + 6\sin x} =$

4. $\dfrac{e^x - e^{3x}}{e^{2x} - e^x}$

2 Linear Disfunction Disorder (LDD)

Much like in the Mad Slasher case, an instinctual desire to simplify as much as possible can lead one astray to a terrible case of LDD, which is an incorrect use of linearity properties. The following are some examples of LDD:

$$\text{I.} \quad (x+9)^2 = x^2 + 9^2 = x^2 + 81$$

$$\text{II.} \quad \sqrt{x^2+64} = \sqrt{x^2} + \sqrt{64} = x + 8$$

$$\text{III.} \quad \frac{x^2}{x^2-4} = \frac{x^2}{x^2} - \frac{x^2}{4} = 1 - \frac{x^2}{4}$$

$$\text{IV.} \quad \ln(x+e) = \ln(x) + \ln(e)$$

$$\text{V.} \quad 2^{x+4} = 2^x + 2^4$$

All of these are easily verified to be incorrect by plugging in some numbers. The correct interpretation for each of these would be the following:

I. $(x+9)^2 = (x+9)(x+9) = x^2 + 9x + 9x + 81 = x^2 + 18x + 81$

II. $\sqrt{x^2+64} =$ (cannot be simplified further)

III. $\dfrac{x^2}{x^2-4} = \dfrac{x^2}{(x+2)(x-2)} =$ (cannot be simplified further)

IV. $\ln(x+e) =$ (cannot be simplified further)

V. $2^{x+4} = 2^x \cdot 2^4$ (by the law of exponents)

Here are some problems for you to try. Simplify the following

1. $(x^2 - 3x)^3 =$
2. $\sqrt{x^4 + 4x^2 + 4} =$
3. $\dfrac{x^2}{x^2 - 4x} =$
4. $5^{2x-3} \cdot 5^9 =$
5. $\dfrac{3^{x+4}}{3^2} =$
6. $\ln[(x \cdot e)^2] + \ln x - 4 =$

3 Notation Mutation (NM)

Few things irk professors more than bad notation when writing mathematics. Students, probably from a lifetime of just "**doing**" math problems, write math like they speak with their friends, using slang and shortening words when convenient. The issue is that when one is writing mathematics one is actually making a well crafted and reasoned argument and using sloppy and incorrect notation is akin to trying to write a formal letter in slang. Imagine you get an unfair ticket for not completely stopping at a stop sign and your defense goes something like :

"Well yo judge I was drivin' and i stopped but the cop didn't see me, so i should be left off the hook".

Compare that to this:

"I respecfully submit this written declaration to the court pursuant to CVC 40902. I plead not guilty to the charge......... CVC 22450 clearly states ....This law does not require a driver to stop *behind* the limit line but *at* the limit line....I believe that a reasonable interpretation of CVC22450 proves my innocence in this case..... (you can fill in the blanks)"

Obviously you will be taken more seriously with the second argument. Ok, now on to the most common notation gaffes made by students regarding equations and expressions!

First, what is the difference between an **expression** and an **equation**? An **equation** is a mathematical sentence that describes the equality of two expressions. In that sense, you can think of an **expression** as a mathematical phrase that describes a numerical value. Here are some examples:

$$\text{Equation:} \quad V = \frac{4}{3}\pi r^3$$

$$\text{Equation:} \quad \sin^2\theta + \cos^2\theta = 1$$

$$\text{Expression:} \quad \sin\theta - 2\cos\theta$$

$$\text{Expression:} \quad x^3 - 4x + 2y$$

Sometimes, when solving an equation, for example, $2x + 4 = 7$, you might solve by writing as follows,

$$\text{I.} \quad 2x + 4 = 7 = 3 = 3/2$$

$$\text{II.} \quad 2x + 4 = 7 \implies 3 \implies 3/2$$

Ok, both of these are examples of equations, but show incorrect use of the equal sign. When reading I, if reading as is, it says that $2x + 4$ is equal to 7, which is then equal to 3, and finally, that 3 is equal to 3/2. It is important when writing to use proper notation so others can understand what you are trying to say. One of the proper ways to solve an equation is to work downwards, making sure each equation is equal to the previous one. This would look like

$$\begin{aligned} 2x + 4 &= 7 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

Now for II, the symbol $\implies$ means "implies". You use this symbol when the next statement is implied from the previous one. So for example, using the previous equation as an example you could write

$$2x + 4 = 7 \implies 2x = 3 \implies x = 3/2$$

because the truth of each statement implies the next statement is also true, but it should never be used as a substitute for the = symbol.

Now on to **expressions.** Expressions may be simplified, but many times students do funny things by treating the numerator and denominator of a rational expression like the two sides of the equal sign in an equation. Here are some examples of this funny behavior:

$$\text{I.} \quad \frac{(x-4)^2}{(x+2)^2} = \frac{(x-4)^{\cancel{2}}}{(x+2)^{\cancel{2}}} = \frac{x-4}{x+2}$$

$$\text{II.} \quad \frac{\sqrt{x^2-4}}{\sqrt{x-2}} = \frac{\left(\sqrt{x^2-4}\right)^2}{\left(\sqrt{x-2}\right)^2} = \frac{x^2-4}{x-2}$$

Both I and II are examples of techniques used when **solving equations** that you cannot do when simplifying rational expressions. For example, when solving an equation such as $(2x-3)^2 = (x+1)^2$ it is entirely appropriate to get the plus or minus square root of both sides to solve the equation. When simplifying an expression, however, there are no two sides to perform the same operation. Everything operation is only done on a single side. One of the proper ways to simplify an expression is to write the original expression on the left side, then work downwards while leaving the left side blank. For example, to simplify the expression $\dfrac{3}{x-2} + \dfrac{5}{x+1}$ you can write:

$$\begin{aligned}
\frac{3}{x-2} + \frac{5}{x+1} &= \frac{3}{x-2} \cdot \frac{(x+1)}{(x+1)} + \frac{5}{x+1} \cdot \frac{(x-2)}{(x-2)} \\
&= \frac{3x+3}{(x-2)(x+1)} + \frac{5x-10}{(x+1)(x-2)} \\
&= \frac{8x-7}{(x-2)(x+1)}
\end{aligned}$$

Notice the difference between solving an equation and simplifying an expression. When we solved the equation the left side always had at least one term from the top to the bottom. When simplifying an expression we leave the left side blank because it is understood that the first expression that we wrote down at the top left is equal to all expressions on the right.

4 Trigonometry Gimmickry (TG)

These are some simple and easily corrected mistakes that are commonly made with trigonometry. Some of the most common mistakes are:

$$\text{I.} \quad \csc x \cdot \tan x = \frac{1}{\cancel{\sin}} \cdot \frac{\cancel{\sin}}{\cos} = \frac{1}{\cos} = \sec$$

$$\text{II.} \quad \cos^2 x = \cos x^2$$

$$\text{III.} \quad \sin^{-1}\left(-\sqrt{3}/2\right) = 4\pi/3, 5\pi/3$$

You might have had some difficulty spotting the error on I, or thought it was a typo, but this is a very common mistake that can sometimes lead to other mistakes. The trigonometric functions are, exactly that, functions. As such every trigonometric functions needs an "argument", or an "input"; otherwise it makes no sense. The correct notation would be:

$$\csc x \cdot \tan x = \frac{1}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{\cos x} = \frac{1}{\cos x} = \sec x$$

It might even be better still to always write the trig functions with a parenthesis enclosing the argument, even when it is only x. The problem above would then look like

$$\csc(x) \cdot \tan(x) = \frac{1}{\cancel{\sin(x)}} \cdot \frac{\cancel{\sin(x)}}{\cos(x)} = \frac{1}{\cos(x)} = \sec(x)$$

Error number II is fixed by simply realizing that $\cos^2 x$ is simply a quick and short way to write $(\cos x)^2$, which is different from $\cos x^2 = \cos\left(x^2\right)$. This might be another good reason to always enclose the inside of the trigonometric functions with a parenthesis.

Now on to III. This is incorrect because although $4\pi/3$ and $5\pi/3$ are angles at which $\sin^{-1} x$ is equal to $-\sqrt{3}/2$, $\sin^{-1} x$ is a function and thus can only have one ouput for every input. Typically, when one is evaluating inverse trigonometric functions, for example

$$\sin^{-1}(-\sqrt{3}/2) =$$

can be interpreted as looking for the angle θ such that $\sin\theta = -1/2$ where $-\pi/2 \le \theta \le \pi/2$. Remember that the range of the inverse functions is restricted as follows

- $f(x) = \sin^{-1} x$ has a domain of $[-1, 1]$ and a range of $[-\pi/2, \pi/2]$
- $f(x) = \cos^{-1} x$ has a domain of $[-1, 1]$ and a range of $[0, \pi]$
- $f(x) = \tan^{-1} x$ has a domain of $(-\infty, \infty)$ and a range of $(-\pi/2, \pi/2)$

So that means that $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$ since $\sin(-\pi/3) = -\sqrt{3}/2$.

Similarly, for example, $\cos^{-1}(-\sqrt{3}/2) = 5\pi/6$ since $\cos(5\pi/6) = -\sqrt{3}/2$ and $5\pi/6$ is within the allowed range of $\cos^{-1} x$.

Here are some more for you to try

1. $\cos^{-1}(-1/2) =$
2. $\sin^{-1}(\sqrt{3}/2) =$
3. $\tan^{-1}(1) =$
4. $\cos^{-1}(-1) =$
5. $\sin^{-1}(-1) =$
6. $\tan^{-1}(1/\sqrt{3}) =$
7. $\sin^{-1}(0) =$
8. $\cos^{-1}(0) =$