MAT122 - Homework 6 SOLUTIONS
Due March 17

1. Consider the function f given by $f(x) = \dfrac{1}{x^2+1}$. Find where f is increasing, decreasing, concave up and concave down.

(a) Begin by using the quotient (or product) rule to show that

$$f'(x) = -\frac{2x}{(x^2+1)^2} \quad \text{and } f''(x) = \frac{6x^2-2}{(x^2+1)^3}$$

You'll probably have to use the chain rule as part of the process. Show your work.

$$f'(x) = \frac{(x^2+1)(0)-(1)(2x)}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2}$$

$$f''(x) = \frac{(x^2+1)^2(-2)-(-2x)(2(x^2+1)2x)}{(x^2+1)^4} = \frac{-2(x^2+1)^2+8x^2(x^2+1)}{(x^2+1)^4} = \frac{6x^2-2}{(x^2+1)^3}$$

(b) Next solve $f'(x)=0$ to find where f' changes sign. Use this to determine where f is increasing or decreasing.

$f'(x) = 0 \implies 2x = 0 \implies x = 0$

Test the interval $(-\infty, 0)$: $f'(-1) = 1/2 \implies f'$ is positive on the interval $(-\infty, 0)$.

Test the interval $(0, \infty)$: $f'(1) = -1/2 \implies f'$ is negative on the interval $(0, \infty)$.

Thus f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

(c) Next solve $f''(x) = 0$ to find where f'' changes sign. Use this to determine where f is concave up or concave down.

$f''(x) = 0 \implies 8x^2 - 2 = 0 \implies x^2 - \dfrac{1}{4} = 0 \implies x = \pm\dfrac{1}{2}$

Test the interval $(-\infty, -\frac{1}{2})$: $f''(-1) = \frac{3}{4} \implies f''$ is positive on the interval $(-\infty, -\frac{1}{2})$.

Test the interval $(-\frac{1}{2}, \frac{1}{2})$: $f''(0) = -2 \implies f''$ is negative on the interval $(-\frac{1}{2}, \frac{1}{2})$.

Test the interval $(\frac{1}{2}, \infty)$: $f''(1) = \frac{3}{4} \implies f''$ is positive on the interval $(\frac{1}{2}, \infty)$.

Thus f is concave up on $(-\infty, -\frac{1}{2})$ and $(\frac{1}{2}, \infty)$ and concave down on $(-\frac{1}{2}, \frac{1}{2})$.

(d) Find the y-intercept, x-intercepts, horizontal and vertical asymptotes**.** Use this information and what you learned about f in parts (a), (b) and (c) to graph $y = f(x)$.

***y*-intercept:**

Set $x = 0$ and solve for y: $f(0) = 1 \Longrightarrow$ y-intercept = 1.

***x*-intercepts:**

Set $y = 0$ and solve for x: $0 = \dfrac{1}{1+x^2}$ has no solutions $\Longrightarrow$ no x-intercepts.

***h*-asymptotes:**

$$\lim_{x\to\infty} \frac{1}{1+x^2} = 0 \Longrightarrow y = 0 \text{ is horizontal asymptote.}$$

$$\lim_{x\to-\infty} \frac{1}{1+x^2} = 0 \Longrightarrow y = 0 \text{ is horizontal asymptote.}$$

***v*-asymptotes:**

For a rational function, this can happen only where the denominator equals 0:
$1 + x^2 = 0$ has no solutions $\Longrightarrow$ no vertical asymptotes.

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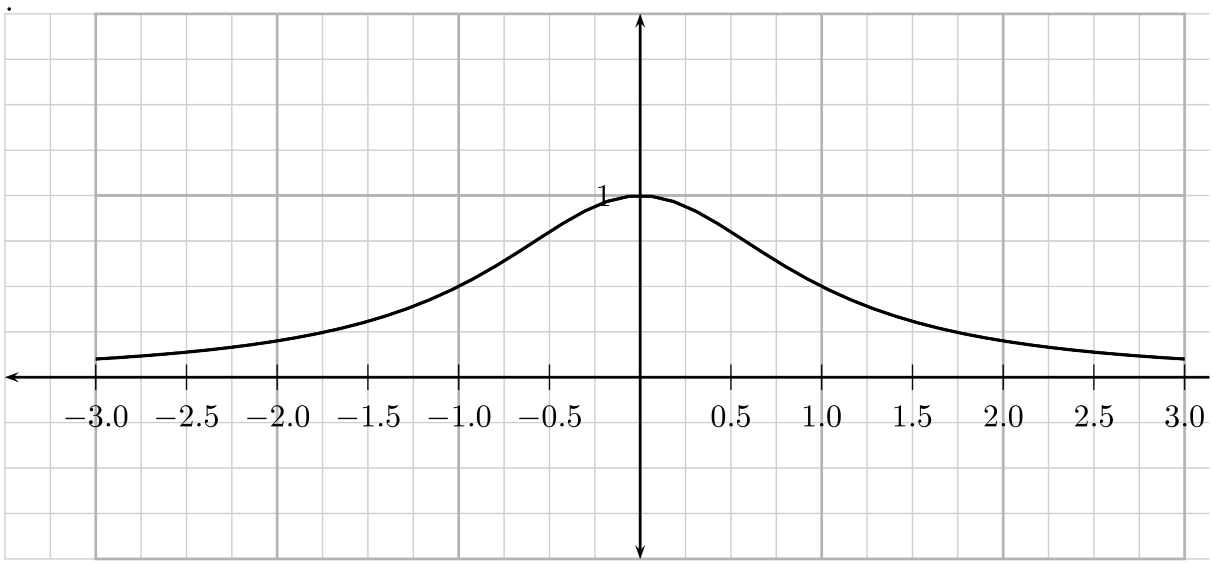

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2. Differentiate

(a) $y = 2x^6 - x + 3$

$y' = 12x^5 - 1$

(b) $f(x) = 5^x + e^x$

$f'(x) = \ln(5) \cdot 5^x + e^x$

(c) $2\sqrt{x} + \sqrt[3]{x}$

$2\sqrt{x} + \sqrt[3]{x} = 2x^{1/2} + x^{1/3}$

$$\begin{aligned}\frac{d}{dx}\left(2x^{1/2} + x^{1/3}\right) &= 2 \cdot \frac{1}{2}x^{-1/2} + \frac{1}{3}x^{-2/3} \\ &= x^{-1/2} + \frac{1}{3}x^{-2/3} \\ &= \frac{1}{\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}}\end{aligned}$$

(d) $\ln(x) + \cos(x)$

$$\frac{d}{dx}\left(\ln(x) + \cos(x)\right) = \frac{1}{x} - \sin(x)$$

3. Use the product rule to differentiate

(a) $\cos(x)e^x$

$$\frac{d}{dx}\left(\cos(x)e^x\right) = -\sin(x)e^x + \cos(x)e^x$$

(b) $x^2 \ln(x)$

$$\begin{aligned}\frac{d}{dx}\left(x^2 \ln(x)\right) &= 2x\ln(x) + x^2\frac{1}{x} \\ &= 2x\ln(x) + x\end{aligned}$$

4. Use the quotient rule to differentiate

(a) $\dfrac{\sin(x)}{x}$

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin(x)}{x}\right) &= \frac{x \cdot \cos(x) - \sin(x) \cdot 1}{x^2} \\ &= \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}\end{aligned}$$

(b) $\dfrac{x}{\ln(x)}$

$$\begin{aligned}\frac{d}{dx}\left(\frac{x}{\ln(x)}\right) &= \frac{\ln(x) \cdot 1 - x \cdot (1/x)}{(\ln(x))^2} \\ &= \frac{\ln(x) - 1}{(\ln(x))^2}\end{aligned}$$

5. Use the chain rule to differentiate

(a) $\sqrt{\sin(x) + x}$

$$\frac{d}{dx}\sqrt{\sin(x) + x} = \frac{1}{2\sqrt{\sin(x) + x}} \cdot (\cos(x) + 1)$$

(b) $e^{\tan(x)}$

$$\frac{d}{dx}e^{\tan(x)} = e^{\tan(x)} \sec^2(x)$$

(c) $[\sin(2x + 1)]^3$

$$\begin{aligned}\frac{d}{dx}[\sin(2x+1)]^3 &= 3[\sin(2x+1)]^2 \cdot \frac{d}{dx}\sin(2x+1) \\ &= 3[\sin(2x+1)]^2 \cdot \cos(2x+1) \cdot 2 \\ &= 6[\sin(2x+1)]^2 \cdot \cos(2x+1)\end{aligned}$$