MAT122
Practice Exam 2

1. Consider the function f given by

$$f(x) = x^{4/3} + \frac{x}{3}$$

Use a linear approximation at $x = 27$ to estimate $f(30)$.

$f'(x) = \frac{4}{3}x^{1/3} + \frac{1}{3}$
$f'(27) = \frac{4}{3} \cdot 27^{1/3} + \frac{1}{3} = \frac{13}{3}$
$f(27) = 27^{4/3} + \frac{27}{3} = 90$

$l(x) = \frac{13}{3}(x - 27) + 90$
$l(30) = \frac{13}{3}(30 - 27) + 90 = 103$

Therefore, $f(30) \cong 103$.

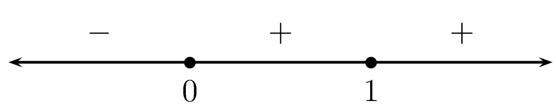
2. Use the first derivative test to find the local minima and maxima of

$$f(x) = 3x^4 - 8x^3 + 6x^2 + 13$$

$f'(x) = 12x^3 - 24x^2 + 12x$
Critical numbers: $f'(x) = 0$ or undefined

$$\begin{aligned} f'(x) &= 12x^3 - 24x^2 + 12x \\ &= 12x(x^2 - 2x + 1) \\ &= 12x(x-1)^2 \end{aligned}$$

We see that $f'(x) = 0$ when $x = 0, 1$.
Sign of f':



We see that f is decreasing on $(-\infty, 0)$ and nondecreasing on $(0, \infty)$. It has a local minimum at $x = 0$.

3. Use the second derivative test to find the local minima, maxima, and inflection points of

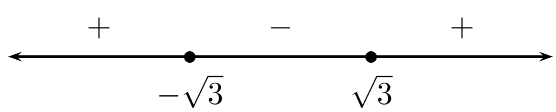
$$f(x) = x^4 - 18x^2$$

$f'(x) = 4x^3 - 36x = 4x(x^2 - 9)$
$f''(x) = 12x^2 - 36 = 12(x^2 - 3)$

<u>Critical numbers</u>: $f'(x) = 0$ or undefined
We see that $f'(x) = 0$ when x = 0, -3, 3.

<u>Second derivative test</u>:
$f''(0) = -36 < 0$. Thus f has a local minimum at x = 0.
$f''(-3) = 72 > 0$. Thus f has a local maximum at x = -3.
$f''(3) = 72 > 0$. Thus f has a local maximum at x = 3.

<u>Inflection points</u>:
$f''(x) = 0$ when $x = \pm\sqrt{3}$.
Sign of f'':



f'' has inflection points at $x = \pm\sqrt{3}$.

4. Consider the curve given by the equation

$$y^2(x + y - 1) - 2x^3 = 0.$$

Write the equation of the tangent line through the point (0,1).

<u>Differentiate Implicitly</u>:

$$2yy'(x + y - 1) + y^2(1 + y') - 6x^2 = 0$$

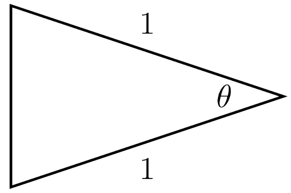
<u>Substitute (0,1)</u>:

$$\begin{aligned} 2 \cdot 1 \cdot y'(0 + 1 - 1) + 1^2(1 + y') - 6 \cdot 0^2 &= 0 \\ 1 + y' &= 0 \\ y' &= -1 \end{aligned}$$

<u>Tangent line:</u>

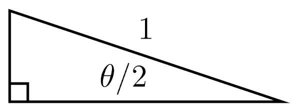
$$\begin{aligned} l(x) &= m(x - x_1) + y_1 \\ &= -1(x - 0) + 1 \\ &= 1 - x \end{aligned}$$

5. Find the angle θ which maximizes the area of the isosceles triangle below.



Hint: First use trigonometry to show that the area can be expressed as $\frac{1}{2}\sin(\theta)$.

Consider the upper half of the triangle:



The length of the sides opposite and adjacent to the given angle are $\sin(\theta/2)$ and $\cos(\theta/2)$, respectively. Therefore the triangle has area

$$\begin{aligned} A &= \sin(\theta/2)\cos(\theta/2) \\ &= \frac{1}{2}\sin(\theta) \end{aligned}$$

where the last line follows from the half-angle formula.

<u>Critical numbers of A</u>: $A' = \frac{d}{d\theta}A = 0$ or undefined

$$A' = \tfrac{1}{2}\cos(\theta)$$

Observe that $A' = 0$ when θ is an odd multiple of $\pi/2$. Since the only feasible values for θ are in the interval $[0, \pi]$, we conclude that A is maximized for $\theta = \pi/2$.

6. Consider a spherical tank of radius 1m which is being filled with water at a rate of 3 m^3/min. If you take take calculus II, you'll learn how to show that the volume of water at height h is given by $\pi(h^2 - \frac{1}{3}h^3)$. At what rate is the height of water increasing at the instant the height of water is $\frac{1}{2}$m?

$$V' = \tfrac{d}{dt}V = \pi(2hh' - h^2h') = \pi hh'(2-h)$$

When $h = \frac{1}{2}$ we have (using $V' = 3$)

$$\begin{aligned} V' &= \pi\frac{1}{2}h'(2 - \frac{1}{2}) \\ 3 &= \frac{3}{4}\pi h' \\ \frac{4}{\pi} &= h' \end{aligned}$$

The height is increasing at a rate of $4/\pi$ m/min.

7. Suppose the spherical tank above began empty and was half full 1 minute later, use the mean value theorem to argue that at some point the water was flowing in at a rate of at least 2 m^3/min.

V is differentiable on the interval [0,1].
Therefore, by the mean value theorem, there is a time *t* in the interval [0,1] such that

$$V'(c) = \frac{V(1) - V(0)}{1 - 0} = \frac{\frac{2}{3}\pi - 0}{1} = \frac{2}{3}\pi$$

Since $\pi > 3$, we see that $V'(c) > 2$ m^3/min.

8. Compute the derivatives of the following expressions. Simplify your answers.

(a) $\arcsin(3x^2)$

$$\frac{d}{dx}\arcsin(3x^2) = \frac{1}{\sqrt{1-(3x^2)^2}} \cdot 6x = \frac{6x}{\sqrt{1-9x^4}}$$

(b) $\ln(\sin(x))$

$$\frac{d}{dx}\ln(\sin(x)) = \frac{1}{\sin(x)} \cdot \cos(x) = \cot(x)$$

(c) $\frac{x-1}{x+1}$

$$\frac{d}{dx}\frac{x-1}{x+1} = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

(d) x^3e^{2x}

$$\frac{d}{dx}(x^3e^{2x}) = 3x^2e^{2x} + x^3e^{2x} \cdot 2 = x^2e^{2x}(3+2x)$$

(e) $5x^7 - 4x - \frac{3}{x} + 8$

$$\frac{d}{dx}(5x^7 - 4x - \frac{3}{x} + 8) = 35x^6 - 4 + \frac{3}{x^2}$$

(f) $\sqrt{x} - \sqrt[3]{x} = x^{1/2} - x^{1/3}$

$$\frac{d}{dx}(x^{1/2} - x^{1/3}) = \frac{1}{2}x^{-1/2} - \frac{1}{3}x^{-2/3}$$