MAT123 Homework 4: Solutions

Calculate the integrals.

7.7.7.

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx = \lim_{t\to 0^+} \Big[x\ln x - x\Big]_t^1 = \lim_{t\to 0^+} [(-1) - (t)] = -1$$

7.7.14.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dz}{z^2+25} &= \lim_{t\to-\infty} \int_t^0 \frac{dz}{z^2+25} + \lim_{t\to\infty} \int_0^t \frac{dz}{z^2+25} \\
&= \lim_{t\to-\infty} \left[\frac{1}{5}\arctan\left(\frac{z}{5}\right)\right]_t^0 + \lim_{t\to\infty} \left[\frac{1}{5}\arctan\left(\frac{z}{5}\right)\right]_0^t \\
&= \lim_{t\to-\infty} \left[-\frac{1}{5}\arctan\left(\frac{t}{5}\right)\right] + \lim_{t\to\infty} \left[\frac{1}{5}\arctan\left(\frac{t}{5}\right)\right] \\
&= -\frac{1}{5}\left(-\frac{\pi}{2}\right) + \frac{1}{5}\left(\frac{\pi}{2}\right) \\
&= \frac{\pi}{10} + \frac{\pi}{10} = \frac{\pi}{5}
\end{aligned}$$

7.7.20.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{\sqrt{x^2+1}}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x^2+1}}\,dx \\
&= \lim_{t\to\infty} \left[\ln\left|x+\sqrt{x^2+1}\right|\right]_1^t \\
&= \lim_{t\to\infty} \left[\ln\left|t+\sqrt{t^2+1}\right| - \ln\left|1+\sqrt{2}\right|\right] \\
&= \infty
\end{aligned}$$

The integral diverges.

7.7.26.

$$\int_1^2 \frac{dx}{x\ln x} = \lim_{t\to 1^+} \int_t^2 \frac{dx}{x\ln x}$$

Let $u = \ln x$. Then $du = \frac{1}{x}dx$ and $\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + K = \ln|\ln x| + K$. Therefore,

$\int_1^2 \frac{dx}{x\ln x} = \lim_{t\to 1^+} \int_t^2 \frac{dx}{x\ln x} = \lim_{t\to 1^+} \Big[\ln|\ln 2| - \ln|\ln t|\Big] = \infty$. The integral diverges.

7.7.30.

$$\begin{aligned}\int_4^\infty \frac{dx}{(x-1)^2} &= \int_4^1 \frac{dx}{(x-1)^2} + \int_1^\infty \frac{dx}{(x-1)^2}\\ &= \lim_{t\to 1^-} \int_4^t \frac{dx}{(x-1)^2} + \lim_{t\to 1^+} \int_t^\infty \frac{dx}{(x-1)^2}\\ &= \lim_{t\to 1^-} \left[-\frac{1}{x-1}\right]_4^t + \lim_{t\to 1^+} \left[-\frac{1}{x-1}\right]_t^\infty\\ &= \lim_{t\to 1^-} \left[-\frac{1}{t-1} + \frac{1}{3}\right] + \lim_{t\to 1^+} \left[0 + \frac{1}{t-1}\right]\end{aligned}$$

Since $\lim_{t\to 1^-} \left[-\frac{1}{t-1} + \frac{1}{3}\right] = \infty$, the integral diverges.