MAT123 Homework 5: Solutions

Decide whether the integral converges or diverges. Explain your reasoning.

7.8.11. $\displaystyle\int_1^\infty \frac{dx}{1+x}$ diverges by the comparison test.

$\displaystyle\frac{1}{1+x} > \frac{1}{x+x} = \frac{1}{2}\frac{1}{x}$, and the integral $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges since the exponent on x is 1.

7.8.13. $\displaystyle\int_5^8 \frac{6}{\sqrt{t-5}}\,dt$ converges.

$\displaystyle\int_5^8 \frac{6}{\sqrt{t-5}}\,dt = \int_0^3 \frac{6}{\sqrt{u}}\,du = 6\int_0^3 \frac{1}{u^{1/2}}\,du$ converges since the exponent 1/2 is less than 1.

7.8.15. $\displaystyle\int_{-1}^5 \frac{dt}{(t+1)^2}$ diverges.

$\displaystyle\int_{-1}^5 \frac{dt}{(t+1)^2} = \int_0^6 \frac{du}{u^2}$ diverges since the exponent 2 is greater than 1.

7.8.17. $\displaystyle\int_1^\infty \frac{du}{u+u^2}$ converges by the comparison test.

$\displaystyle\frac{1}{u+u^2} < \frac{1}{u^2}$ and $\displaystyle\int_1^\infty \frac{1}{u^2}\,du$ converges since the exponent 2 is greater than 1.

7.8.19. $\displaystyle\int_2^\infty \frac{d\theta}{\sqrt{\theta^3+1}}$ converges by the comparison test.

$\displaystyle\frac{1}{\sqrt{\theta^3+1}} < \frac{1}{\sqrt{\theta^3}} = \frac{1}{\theta^{3/2}}$, and $\displaystyle\int_2^\infty \frac{d\theta}{\theta^{3/2}}$ converges since the exponent 3/2 is greater than 1.

7.8.21. $\displaystyle\int_0^\infty \frac{dy}{1+e^y}$ converges by the comparison test.

$\displaystyle\frac{1}{1+e^y} < \frac{1}{e^y} = e^{-y}$, and $\displaystyle\int_0^\infty e^{-y}\,dy$ converges since $e^{-y} = e^{-ay}$ has $a = 1 > 0$.

7.8.23. $\displaystyle\int_0^\infty \frac{dz}{e^z+2^z}$ converges by the comparison test.

$\displaystyle\frac{1}{e^z+2^z} < \frac{1}{e^z} = e^{-z}$, and $\displaystyle\int_0^\infty e^{-z}\,dz$ converges since $e^{-z} = e^{-az}$ has $a = 1 > 0$.

7.8.25. $\int_4^\infty \frac{3+\sin\alpha}{\alpha}\,d\alpha$ diverges by the comparison test.

$\frac{3+\sin\alpha}{\alpha} > \frac{2}{\alpha}$ since $\sin\alpha > -1$. $\int_4^\infty \frac{2}{\alpha}\,d\alpha = 2\int_4^\infty \frac{1}{\alpha}\,d\alpha$ diverges since the exponent on α is not greater than 1.

7.8.31. For what values of p does the integral converge or diverge?

$\int_1^2 \frac{dx}{x(\ln x)^p} = \int_0^{\ln 2} \frac{1}{u^p}\,du.$

This converges when $p < 1$ and diverges when $p \geq 1$.

7.8.33. $\int_1^\infty \frac{dx}{x^5(e^{1/x}-1)}$

(a) **Explain why a graph of the tangent line to e^t at *t* = 0 tells us that for all *t***

$$1 + t \leq e^t.$$

$y = 1 + t$ is the equation of the tangent line to $y = e^t$ at the point (0,1). $y = e^t$ is concave up so its graph must lie above the graph of its tangent line. Therefore, $1 + t \leq e^t$.

(b) **Substituting $t = 1/x$, show that for all $x \neq 0$**

$$e^{1/x} - 1 > \frac{1}{x}$$

Substituting $t = 1/x$ gives $1 + \frac{1}{x} \leq e^{1/x}$ (for $x \neq 0$). Subtract 1 from both sides to get

$$e^{1/x} - 1 > \frac{1}{x}$$

(c) **Use the comparison test to show that the original integral converges.**

$\frac{1}{x^5(e^{1/x}-1)} < \frac{1}{x^5(1/x)} = \frac{1}{x^4}$ when $x \geq 1$, and $\int_1^\infty \frac{1}{x^4}\,dx$ converges since the exponent 4 is greater than 1. Therefore, by the comparison test, $\int_1^\infty \frac{dx}{x^5(e^{1/x}-1)}$ converges.