Homework 6 Solutions

8.1: 3,7,8,9,12,13
8.2: 14,32,33,54

8.1.3.
Let *w* denote the width of the approximating piece at hight h. Then using what we know of similar triangles,

$$\tfrac{w}{5-h} = \tfrac{3}{5} \implies w = \tfrac{3}{5}(5-h).$$

The area (width · height) of the approximating piece is given by

$$\Delta A = \tfrac{3}{5}(5-h)\Delta h.$$

An approximation to the total area is given by the Riemann sum

$$\sum_{i=1}^{n} \tfrac{3}{5}(5-h_i)\Delta h.$$

The actual area is given by the integral

$$\int_0^5 \frac{3}{5}(5-h)\,dh = \frac{3}{5}\left[5h - \frac{1}{2}h^2\right]_0^5 = \frac{3}{5}\cdot\frac{25}{2} = \frac{15}{2}.$$

8.1.7.
The width of the approximating piece at height *y* is given by the *x*-coordinate of the rightmost curve, $x = y$, minus the *x*-coordinate of the leftmost curve, $x = y^2$:

$$y - y^2$$

The area (width · height) of the approximating piece is given by

$$\Delta A = (y - y^2)\Delta y.$$

An approximation to the total area is given by the Riemann sum

$$\sum_{i=1}^{n} (y_i - y_i^2)\Delta y.$$

The actual area is given by the integral

$$\int_0^1 (y - y^2)\,dy = \left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

We integrate from 0 to 1 since these are the *y*-values of the points where the curves intersect. They are found by substituting $x = y$ into the equation $y = \sqrt{x}$ and solving for *y*.

8.1.8.
The height of the approximating piece at *x* is given by the *y*-coordinate of the upper curve, $3x + y = 6$, minus the *y*-coordinate of the lower curve, $y = x^2 - 4$:

$$(6 - 3x) - (x^2 - 4) = 10 - 3x - x^2$$

The area (width · height) of the approximating piece is given by

$$\Delta A = (10 - 3x - x^2)\Delta x.$$

An approximation to the total area is given by the Riemann sum

$$\sum_{i=1}^{n}(10 - 3x_i - x_i^2)\Delta x.$$

The actual area is given by the integral

$$\int_0^2 (10 - 3x - x^2)\,dx = \left[10x - \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^2 = 20 - 6 - \frac{8}{3} = \frac{34}{3}.$$

We integrate to 3 since this is the *x*-value of the point where the curves intersect. It is found by substituting $y = 6 - 3x$ into the equation $y = x^2 - 4$ and solving for *x*.

8.1.9.
The volume of the approximating piece at *x* is given by multiplying the area of a cross section (a circle of radius 2cm) by its thickness Δx:

$$\Delta V = 4\pi \cdot \Delta x$$

An approximation to the total volume is given by the Riemann sum

$$\sum_{i=1}^{n} 4\pi\Delta x.$$

The actual volume is given by the integral

$$\int_0^9 4\pi\,dx = 4\pi x\Big|_0^9 = 36\pi \text{ cm}^3$$

We integrate to 9 since the cylinder has length 9cm.

8.1.12.
The width of a cross section at height *y* is given by subtracting the two *x*-coordinates of the corresponding points on the circle of radius 7m:

$$(\sqrt{49-y^2}) - (-\sqrt{49-y^2}) = 2\sqrt{49-y^2}$$

The volume of the approximating piece at *y* is given by multiplying the width of a cross section by its length (10m) and by its thickness Δy.

$$\Delta V = 2\sqrt{49-y^2}\cdot 10\cdot \Delta y$$

An approximation to the total volume is given by the Riemann sum

$$\sum_{i=1}^{n} 20\sqrt{49-y^2}\,\Delta y.$$

The actual volume is given by the integral

$$\int_0^7 20\sqrt{49-y^2}\,dy = 20\left[\frac{1}{2}\left(y\sqrt{49-y^2}+49\arcsin\frac{y}{7}\right)\right]_0^7 = 20\cdot 49\frac{\pi}{2}\pi = 490\pi \text{ m}^3$$

We integrate to 7 since the object has height 7m.

8.1.13.
The area of a cross section (a circle) at height *y* is given by πr^2 where *r* is its radius. *r* is given as the *x*-coordinate of a circle centered at the origin with radius 5mm. Thus $r = \sqrt{25-y^2}$ and so the area of the cross section is given by

$$\pi\left(\sqrt{25-y^2}\right)^2 = \pi(25-y^2)$$

The volume of the approximating piece at *y* is given by multiplying the area of the cross section at *y* by its thickness Δy.

$$\Delta V = \pi(25-y^2)\cdot\Delta y$$

An approximation to the total volume is given by the Riemann sum

$$\sum_{i=1}^{n} \pi(25-y^2)\,\Delta y.$$

The actual volume is given by the integral

$$\int_0^5 \pi(25-y^2)\,dy = \pi\left[25y-\frac{1}{3}y^3\right]_0^5 = \pi\cdot\left(125-\frac{125}{3}\right) = \frac{250}{3}\pi \text{ mm}^3$$

We integrate to 5 since the object has height 5mm.

8.2.14.

$f(x) = x^{3/2} \implies f'(x) = \frac{3}{2}x^{1/2} \implies [f'(x)]^2 = \frac{9}{4}x$

$$\int_0^2 \sqrt{1+\frac{9}{4}x}\,dx = \left[\frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\right]_0^2 = \frac{44}{27} - \frac{8}{27} = \frac{36}{27} = \frac{4}{3}$$

8.2.32.

By the method of cylindrical shells:

$$V = \int_0^1 2\pi x(1-x^2)\,dx = 2\pi\int_0^1 (x-x^3)\,dx = 2\pi\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = 2\pi\cdot\frac{1}{4} = \frac{\pi}{2}$$

We integrate from 0 to 1 since the *y*-axis is the line $x = 0$ and the curves $y = x^2$ and $y = 1$ meet when $x = 1$.

By the washer method:

The washer method requires us to write the curve $y = x^2$ as a function of *y*, $x = \sqrt{y}$. Then

$$V = \int_0^1 \pi(\sqrt{y})^2\,dy = \pi\int_0^1 y\,dy = \pi\left[\frac{1}{2}y^2\right]_0^1 = \frac{\pi}{2}$$

We integrate from 0 to 1 since the curves $x = \sqrt{y}$ meets the *y*-axis when $y = 0$ and the region is bounded by the line $y = 1$.

8.2.33.

By the washer method:

$$V = \int_0^1 \pi(1)^2 - \pi\left(x^2\right)^2\,dy = \pi\int_0^1 1 - x^4\,dy = \pi\left[x - \frac{1}{5}x^5\right]_0^1 = \frac{4\pi}{5}$$

We integrate from 0 to 1 since the *y*-axis is the line $x = 0$ and the curves $y = x^2$ and $y = 1$ meet when $x = 1$.

8.2.54.
The ends are located at $x = -b$ and $x = b$. Therefore the distance between them is $b - (-b) = 2b$. It is given that the length of the chain is 10m and we know this is given by the arc length formula:

$$\int_{-b}^{b} \sqrt{1 + [f'(x)]^2}\, dx$$

We have

$$f(x) = \frac{1}{2}\left(e^x + e^{-x}\right) \implies f'(x) = \frac{1}{2}\left(e^x - e^{-x}\right) \implies [f'(x)]^2 = \frac{1}{4}\left(e^x - e^{-x}\right)^2$$
$$= \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right)$$

Therefore*

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right) \\ &= 1 + \frac{1}{4}e^{2x} + \frac{1}{4}e^{-2x} - \frac{1}{2} \\ &= \frac{1}{4}\left(e^{2x} + e^{-2x} + 2\right) \\ &= [f(x)]^2 \end{aligned}$$

So

$$\begin{aligned} \int_{-b}^{b} \sqrt{1 + [f'(x)]^2}\, dx &= \int_{-b}^{b} \sqrt{[f(x)]^2}\, dx \\ &= \int_{-b}^{b} |f(x)|\, dx \\ &= \int_{-b}^{b} \frac{1}{2}\left(e^x + e^{-x}\right)\, dx \\ &= \frac{1}{2}\left[e^x - e^{-x}\right]_{-b}^{b} \\ &= \frac{1}{2}\left[\left(e^b - e^{-b}\right) - \left(e^{-b} - e^b\right)\right] \\ &= e^b - e^{-b} \end{aligned}$$

The chain has length 10m. Therefore

$$e^b - e^{-b} = 10$$

Put $u = e^b$. Then

$$u - u^{-1} = 10 \implies u^2 - 1 = 10u \implies u^2 - 10u - 1 = 0$$

The quadratic formula gives $u = \frac{10 \pm \sqrt{104}}{2} = 5 \pm \sqrt{26}$. We take the positive solution $u = 5 + \sqrt{26}$ and solve $u = e^b$ for b:

$$b = \ln\left(5 + \sqrt{26}\right)$$

As observed earlier the distance between the ends is $2b$, so the distance is $2\ln\left(5 + \sqrt{26}\right)$.

* In terms of the hyperbolic trig functions, this is a combination of the relationships

$$\frac{d}{dx}\cosh(x) = \sinh(x) \text{ and } 1 + \sinh^2(x) = \cosh^2(x)$$