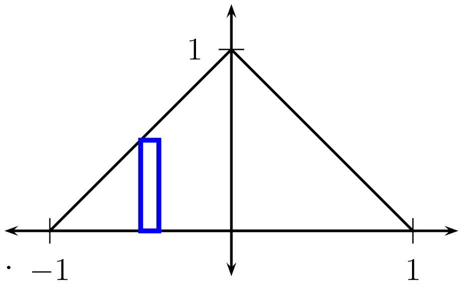
Homework 7 Solutions

8.4.10.



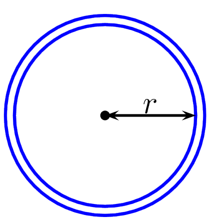
Mass of blue rectangle at x:

$$\delta(x)\cdot dA = \begin{cases} (1+x)\cdot(1+x)dx & \text{for } x<0 \\ (1+x)\cdot(1-x)dx & \text{for } x>0 \end{cases}$$

Total mass =

$$\int_{-1}^{0}(1+x)^2dx + \int_0^1(1-x^2)dx = \left[\frac{1}{3}(1+x)^3\right]_{-1}^{0} + \left[x-\frac{1}{3}x^3\right]_0^1$$
$$= \frac{1}{3}+\frac{2}{3}$$
$$= 1$$

8.4.11.



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Population density: $\delta(r) = 10000(3-r)$

(a) density becomes 0 when $r=3 \Longrightarrow$ radius of city = 3 miles

(b) Area of blue ring: $dA = 2\pi r\cdot dr$
Population within blue ring: $\delta(r)\cdot dA = 10000(3-r)\cdot 2\pi r\cdot dr$

Total population = $\int_0^3 10000(3-r)\cdot 2\pi r\cdot dr = 20000\pi\left[\frac{3}{2}r^2-\frac{1}{3}r^3\right]_0^3 = 20000\pi\left[\frac{27}{2}-9\right]$

= 90000π

8.4.20.

Total Mass = $\int_0^3 1 + x^2 dx = \left[x + \frac{1}{3}x^3\right]_0^3 = 12$ grams

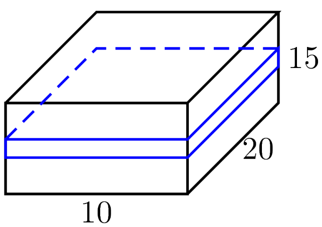
Center of Mass = $\frac{1}{12}\int_0^3 (1+x^2)x\,dx = \frac{1}{12}\left[\frac{1}{2}x^2 + \frac{1}{4}x^4\right]_0^3 = \frac{99}{48}$

8.5.7.

Let y denote height above bottom of lake.
Force required to lift anchor: 100 lb
Force required to lift chain when anchor is at height y above bottom of lake: $3(25-y)$
Total force required to lift anchor and chain: $100 + 3(25-y)$
Work to displace anchor amount dy: $dW = (100 + 3(25-y)) \cdot dy$

Total work = $\int_0^{25} (100 + 3(25-y))\,dy$ = 3437.5 ft-lbs

8.5.11.



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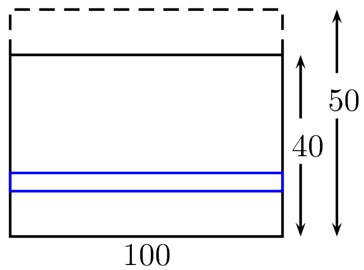
Density of water: 62.4 lb / ft^3
Let y denote height of slab above base of tank.
Volume of slab: $dV = 10 \cdot 20 \cdot dy = 200\,dy$
Force required to lift slab: $62.4 \cdot 200\,dy = 12480\,dy$
Work required to move slab out of tank: $dW = 12480\,dy \cdot (15-y) = 12480(15-y)\,dy$

Total work = $\int_0^{15} 12480(15-y)\,dy$ = 1404000 ft-lbs

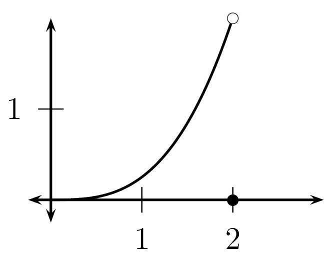
8.5.23.



Density of water: 62.4 lb / ft^3
Let y denote the height of a slice of water.
Area of slice = $100 \cdot dy$
Pressure of water at height y: $62.4 \cdot (40 - y)$ (see p.428 on British units)
Force exerted by water at height y: Pressure · Area = $62.4(40 - y) \cdot 100 \cdot dy = 6240(40 - y)\,dy$

Total force: $\int_0^{40} 6240(40 - y)\,dy = 4992000$ ft-lbs

8.8.5.



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(a) P(1.5 < x < 2.0) = $\int_{1.5}^{2.0} \frac{x^3}{4}\,dx = \left[\frac{1}{16}x^4\right]_{1.5}^{2.0} \cong 1 - 0.3164 \cong 0.6836$

(b) mean = $\int_0^2 x \cdot \frac{x^3}{4}\,dx = \left[\frac{1}{20}x^5\right]_0^2 = \frac{32}{20} = 1.6$

(c) median = T such that $\int_0^T \frac{x^3}{4}\,dx = \frac{1}{2}$.

$$\int_0^T \frac{x^3}{4}\,dx = \left[\frac{1}{16}x^4\right]_0^T = \frac{T^4}{16} = \frac{1}{2} \implies T^4 = 8 \implies T = \sqrt[4]{8}$$

8.8.9.

(a) $p(x) = \dfrac{1}{15\sqrt{2\pi}}\, e^{-(x-100)^2/450}$

(b) P(115 < x < 120) $= \displaystyle\int_{115}^{120} \frac{1}{15\sqrt{2\pi}}\, e^{-(x-100)^2/450}\, dx \cong 0.0674$ (Simpson's rule can be used to make such approximations.)

8.8.10.

The speeds of cars are approximately normally distributed with mean $\mu = 58$ km/hr and standard deviation $\sigma = 4$ km/hr.

(a) P$(60 < v < 65) \cong \displaystyle\int_{60}^{65} \frac{1}{4\sqrt{2\pi}}\, e^{-(x-58)^2/32}\, dx \cong 0.2685$

(b) P$(v < 52) \cong \displaystyle\int_{-\infty}^{52} \frac{1}{4\sqrt{2\pi}}\, e^{-(x-58)^2/32}\, dx \cong 0.0668$

8.8.11.

(a)

$$\frac{d}{dx} e^{-(x-\mu)^2/(2\sigma^2)} = \frac{-(x-\mu)}{\sigma^2} e^{-(x-\mu)^2/(2\sigma^2)}$$

$$\begin{aligned}\frac{d^2}{dx^2} e^{-(x-\mu)^2/(2\sigma^2)} &= \frac{-(x-\mu)}{\sigma^2} \cdot \frac{-(x-\mu)}{\sigma^2} e^{-(x-\mu)^2/(2\sigma^2)} + \frac{-1}{\sigma^2} e^{-(x-\mu)^2/(2\sigma^2)} \\ &= \frac{(x-\mu)^2}{\sigma^4} e^{-(x-\mu)^2/(2\sigma^2)} - \frac{1}{\sigma^2} e^{-(x-\mu)^2/(2\sigma^2)} \\ &= \left(\frac{(x-\mu)^2}{\sigma^4} - \frac{1}{\sigma^2}\right) e^{-(x-\mu)^2/(2\sigma^2)}\end{aligned}$$

$$p'(x) = \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} = 0 \implies x = \mu$$

$$p''(x) = \frac{1}{\sigma\sqrt{2\pi}} \left(\frac{(x-\mu)^2}{\sigma^4} - \frac{1}{\sigma^2}\right) e^{-(x-\mu)^2/(2\sigma^2)} < 0 \text{ when } x = \mu$$

Therefore, by the second derivative test, $p(x)$ has a maximum at μ.

(b) $p''(x)$ changes sign where $\dfrac{(x-\mu)^2}{\sigma^4} - \dfrac{1}{\sigma^2}$ changes sign, since the other factors are always positive.

$$\frac{(x-\mu)^2}{\sigma^4} - \frac{1}{\sigma^2} = \frac{(x-\mu)^2 - \sigma^2}{\sigma^4}$$

This is negative when $\mu - \sigma < x < \mu + \sigma$ and positive otherwise. Therefore p has inflection points at $\mu - \sigma$ and $\mu + \sigma$.

(c) μ determines the center of the distribution and σ determines how spread out it is.