MAT123
Exam 3 Practice Problems
Updated May 5

Don't panic. The actual final exam will not be this long. But do study hard!!

1. Use the comparison test to determine whether the following improper integrals converge or diverge.

$$\int_0^5 \frac{2}{\sqrt{x+x^2}}\,dx$$

$$\frac{1}{\sqrt{x+x^2}} \le \frac{1}{\sqrt{x}}$$

$\displaystyle\int_0^5 \frac{1}{\sqrt{x}}\,dx$ converges since the exponent 1/2 is less than 1. Therefore the given integral converges by the comparison test.

$$\int_1^\infty \frac{e^{-x}}{\sqrt{x}+x^2}\,dx$$

$$\frac{e^{-x}}{\sqrt{x}+x^2} \le e^{-x}$$

$\displaystyle\int_1^\infty e^{-x}\,dx$ converges since the exponent is negative. Therefore the given integral converges by the comparison test.

$$\int_0^1 \frac{1}{\sqrt{9x^2-x^3}}\,dx$$

$$\frac{1}{\sqrt{9x^2-x^3}} \ge \frac{1}{\sqrt{9x^2}} = \frac{1}{3x} = \frac{1}{3}\cdot\frac{1}{x}$$

$\displaystyle\int_0^1 \frac{1}{x}\,dx$ diverges since the exponent 1 is not less than 1. Therefore the given integral converges by the comparison test.

2. Find the volume obtained by rotating the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$
(a) about the *x*-axis.

$$\pi\int_0^1 \left(\sqrt{x}\right)^2 - \left(x^2\right)^2\,dx = \pi\int_0^1 x - x^4\,dx = \frac{3\pi}{10}$$

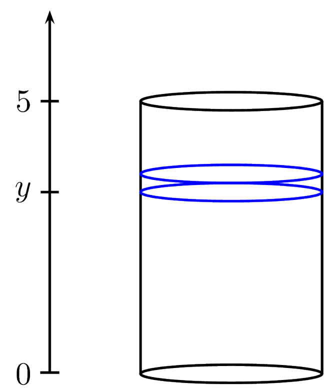
(b) about the *y*-axis.

$$2\pi\int_0^1 x\left(\sqrt{x}-x^2\right)\,dx = 2\pi\int_0^1 x^{3/2} - x^3\,dx = \frac{3\pi}{10}$$

3. Find the area bounded between the curves $3 - x = 3y^2$ and $x + 5 = 5y^4$. They intersect at (0,1) and (0,-1).

$$\int_{-1}^{1} (3 - 3y^2) - (5y^4 - 5)\, dy = \int_{-1}^{1} 8 - 3y^2 - 5y^4\, dy = 12$$

4. A cylindrical tank with radius 2ft and height 5ft is standing on its circular base and is filled with a fluid having density of 10 lb/ft^3. Compute the force exerted by the fluid on the side of the tank.



Area of slice: $dA = 2\pi r \cdot dy = 4\pi\, dy$
Pressure at height y: $P = 10 \cdot (5 - y)$
Force on slice: $dF = P \cdot dA = 40\pi(5 - y)dy$

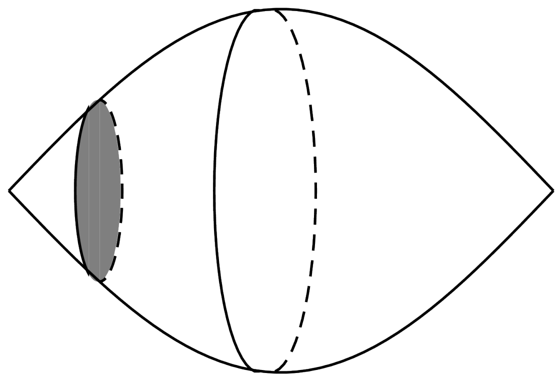
Total force: $F = \displaystyle\int_0^5 dF(y) = \int_0^5 40\pi(5 - y)\, dy = 500\pi$

5. Write an integral which gives the arc length of the curve given by $y = \ln(x)$ between the points (1,0) and (e,1). Do not attempt to evaluate the integral.

$$\frac{dy}{dx} = \frac{1}{x}$$

$$L = \int_1^e \sqrt{1 + \left(\frac{1}{x}\right)^2}\, dx$$

6. Find the volume of region below whose cross-sectional area (the area of the shaded region) at distance x from one end is given by $A(x) = \sin(x)$.



Total area:

$$A = \int_0^\pi dA = \int_0^\pi \sin(x)\,dx = 2$$

7. Find the limit of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+1}}$.

$$\frac{(-2)^n}{3^{n+1}} = \frac{1}{3} \cdot \left(\frac{-2}{3}\right)^n$$

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n = \frac{1}{3}\left(\sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n - 1\right) = \frac{1}{3}\left(\frac{1}{1+\frac{2}{3}} - 1\right) = \frac{-2}{15}$$

8. Explain why the following series converge or diverge:

(a) $\sum_{n=0}^{\infty} \sqrt{\frac{n+1}{n}}$

$\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \sqrt{\lim_{n\to\infty} \frac{n+1}{n}} = 1.$ Therefore the series diverges by the divergence test.

(b) $\sum_{n=0}^{\infty} \left(\frac{-4}{3+n}\right)^n$

$\left(\frac{-4}{3+n}\right)^n = (-1)^n \left(\frac{4}{3+n}\right)^n$

$\left(\frac{4}{3+n}\right)^n$ decreases and approaches 0 as $n \to \infty$. Thus the series converges by the alternating series test.

(c) $\sum_{n=0}^{\infty} \frac{5^n}{n!}$

$\frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1}$ and $\lim_{n\to\infty} \frac{5}{n+1} = 0$. The limit is less than 1. Therefore, by the ratio test, the series converges.

9. Suppose a function f gives rise to the following table of values:

x	$f^{(0)}(x)$	$f^{(1)}(x)$	$f^{(2)}(x)$	$f^{(3)}(x)$
0	2	3	0	-1
2	5	0	-4	1

(a) Give the 3rd order Taylor approximation to f centered at 0.

$2 + 3x - \frac{1}{6}x^3$

(b) Give the 3rd order Taylor approximation to f centered at 2.

$5 - 2(x-2)^2 + \frac{1}{6}(x-2)^3$

10. Use the fact that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$ to write the Taylor series for $\arctan(x)$ centered at 0.

$$\frac{d}{dx}\arctan(x) = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\arctan(x) = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}x^{2n+1} + C$$

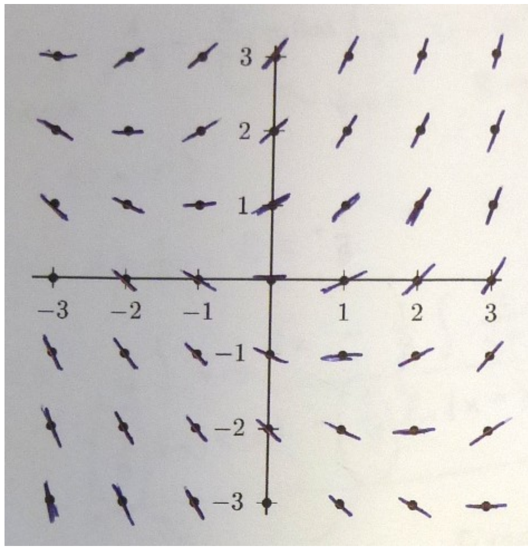
To determine C, let $x = 0$. Then

$$\arctan(0) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}0^{2n+1} + C \implies 0 = C$$

Therefore, $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}x^{2n+1}.$

11. Graph the slope field for $y' = \frac{x+y}{2}$ using the grid below. Draw one line segment through each grid point.

Use Euler's Method with step size 1 to estimate $y(2)$ where y is the solution which satisfies $y(-2) = 1$.



x	y	y'
-2	1	-1/2
-1	1/2	-1/4
0	1/4	1/8
1	3/8	11/16
2	17/16	

$y(2) \approx 17/16.$

12. Compute the numerical value of $\displaystyle\int_3^{\infty} \frac{-2x}{(1+x^2)^2}\,dx.$

Let $u = 1 + x^2$. Then $du = 2x\,dx$. So

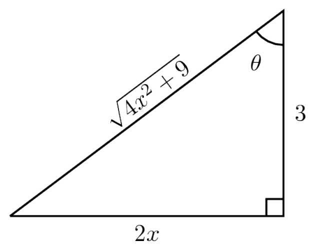
$$\int_3^{\infty} \frac{-2x}{(1+x^2)^2}\,dx = \int_{10}^{\infty} \frac{-1}{u^2}\,du = -\frac{1}{10}$$

13. Use the fact that $\displaystyle\frac{1}{x^2 - 3x - 10} = \frac{A}{x-5} + \frac{B}{x+2}$ to find $\displaystyle\int \frac{1}{x^2-3x-10}\,dx$. You must determine the values of A and B.

$1 = A(x+2) + B(x-5) \implies A = \frac{1}{7}, B = -\frac{1}{7}$

$$\int \frac{1}{x^2-3x-10}\,dx = \frac{1}{7}\int \frac{1}{x-5}\,dx - \frac{1}{7}\int \frac{1}{x+2}\,dx = \frac{1}{7}\ln(x-5) - \frac{1}{7}\ln(x+2) + C$$

14. Use the substitution suggested by the triangle below to compute $\int \frac{9}{4x^2+9}dx$.



$$\cos(\theta) = \frac{3}{\sqrt{4x^2+9}} \implies \cos^2(\theta) = \frac{9}{4x^2+9}$$

$$\tan(\theta) = \frac{2x}{3} \implies \sec^2(\theta)d\theta = \frac{2}{3}dx$$

$$\int \frac{9}{4x^2+9}dx = \int \cos^2(\theta)\frac{3}{2}\sec^2(\theta)\,d\theta = \int \frac{3}{2}\,d\theta = \frac{3}{2}\theta + C = \frac{3}{2}\arctan\left(\frac{2x}{3}\right) + C$$

15. Use the method of separation of variables to solve the differential equation $y' = x + xy$.

$$\begin{aligned} \frac{dy}{dx} = x(1+y) &\implies \frac{dy}{1+y} = xdx \\ &\implies \int \frac{dy}{1+y} = \int xdx \\ &\implies \ln(1+y) = \frac{1}{2}x^2 + C \\ &\implies 1+y = Ae^{\frac{1}{2}x^2} \\ &\implies y = Ae^{\frac{1}{2}x^2} - 1 \end{aligned}$$

16. Verify that $y = Ae^x + Be^{2x}$ is a solution to $y'' - 3y' + 2y = 0$. Find *A* and *B* so that $y(0) = 1$ and $y'(0) = 3$.

$$y' = Ae^x + 2Be^{2x}$$
$$y'' = Ae^x + 4Be^{2x}$$

$$\begin{aligned} y'' - 2y' + 2y &= (Ae^x + 4Be^{2x}) - 3(Ae^x + 2Be^{2x}) + 2(Ae^x + Be^{2x}) \\ &= Ae^x + 4Be^{2x} - 3Ae^x - 6Be^{2x} + 2Ae^x + 2Be^{2x} \\ &= 0 \end{aligned}$$

$y(0) = 1 \implies A + B = 1$ and $y'(0) = 1 \implies A + 2B = 3$.
Solving this system gives *A* = -1 and *B* = 2, so $y = 2e^{2x} - e^x$ is the particular solution.

17. Use integration by parts twice to compute $\int x^2 e^x \, dx$.

$$\begin{aligned}
\int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx \\
&= x^2 e^x - \left(2x e^x - \int 2e^x \, dx \right) \\
&= x^2 e^x - 2x e^x + 2e^x + C \\
&= \left(x^2 - 2x + 2 \right) e^x + C
\end{aligned}$$